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On Fractal Modeling in Astrophysics: The Effect of Lacunarity on the Convergence of Algorithms for Scaling Exponents.

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Abstract. Fractals and multifractals are used to model hierarchical, inhomogeneous structures in several areas of astrophysics, notably the distribution of matter at various scales in the universe. Current analysis techniques used to assert fractality or multifractality and extract scaling exponents from astrophysical data, however, have significant limitations and caveats. It is pointed out that some of the difficulties regarding the convergence rates of algorithms used to determine scaling exponents for a fractal or multifractal, are intrinsically related to its texture, in particular, to its *lacunarity*. A novel approach to characterize prefactors of cover functions, in particular, lacunarity, based on the formalism of *regular variation (in the sense of Karamata)* is proposed. This approach allows deriving bounds on convergence rates for scaling exponent algorithms and may provide more precise characterizations for fractal-like objects of interest for astrophysics. An application of regular variation based fractal modeling to apollonian packing, which can be useful in cosmic voids morphology modeling, is also suggested.

1. Introduction

Structures with fractal characteristics are important for astrophysics. Fractal-like structures may be created whenever nonlinear dynamical mechanisms are at work. Such mechanisms may generate inhomogeneity on a large range of physical scales and "incomplete space filling." For example, certain aspects of the distribution of matter in the universe are well described by fractals at small and medium scales (distribution of galaxy number counts, microwave background fluctuations, etc.) (Borgani 1995). However, the problem of a crossover from "fractality" to "homogeneity" at large scales, as required by the cosmological principle (Peebles 1993), raises some questions related to formal aspects of the fractal modeling involved and the ways in which the data are analyzed (Labini et al. 1996). At this point in time, it is recognized that fractal models with pure scale invariance are not sufficient to describe the morphology of large scale structure. In addition, the details of the mentioned crossover, if it exists, are open. In fact, it was pointed out recently (Mandelbrot 1995) that the research regarding the large scale morphology can benefit from a better understanding of the finer morphological features of fractals, namely, of their texture, and in particular, their *lacunarity*.

2. Lacunarity of a Fractal Structure

For a fractal, the fractal dimension alone is an incomplete description of the degree and nature of space filling. One can generate fractals with exactly the same fractal dimension and very different space filling structure or lacunarity. In a family of fractals with the same fractal dimension, the lower the “lacunarity” of a fractal the closer its appearance is to “homogeneity.” Thus, as pointed out by Mandelbrot, large scale homogeneity may be mimicked by a fractal structure with low lacunarity (Mandelbrot 1983, 1995). Moreover, texture and even the fractal dimension can change with scale (a structure can be fractal without being scale invariant). The existence of “transient fractals” (for which the “dimension” changes with scale) is known in the fractal literature.

In principle, the concept of lacunarity should capture quantitative aspects related to the distribution of magnitudes of *holes* (“empty” spaces) in a fractal structure. Mandelbrot pointed out that there is no unique all-encompassing way to formally define a measure of lacunarity, and proposed to characterize texture properties of fractals using the behavior of their cover functions. In particular, the *prefactor oscillations of a Minkowski-Bouligand cover of a fractal*, can be used to describe *shell lacunarity*. However, save for some particular classes of mathematical fractals (e.g., Theiler 1988; Mandelbrot 1995), the properties of cover functions and their prefactors are poorly understood and a systematic approach to their study is lacking. Moreover, some cover functions may even be nondifferentiable and the presence of high lacunarity, has the effect of degrading the convergence properties of data analysis algorithms used to assert fractality and extract scaling exponents.

3. Using the Regular Variation Formalism for Fractal Texture

The aim of this note is to point out that the use of the mathematical Theory of Regular Variation (in the sense of Karamata) (RVT) and its extensions (de Haan theory) (Bingham et al. 1987), provides a formal framework to study more systematically the properties of cover function prefactors for fractal-like structures and thus, of fractal texture, including lacunarity. Some of the benefits of the RVT based approach are a) more precise characterizations of a variety of morphological features of fractal-like structures, not captured by fractal dimension, in particular, a taxonomy of fractal-like structures based on the “roughness” of their cover functions, b) means to quantify differences among fractals with the same fractal dimensions, in particular, a rigorous approach to shell lacunarity, c) bounds on rates of convergence for some basic algorithms used to evaluate fractal dimensions and other scaling exponents, and a base to develop novel algorithms for this purpose, and d) means to detect and characterize scale dependence using “fine” properties of cover function. The texture characterization methodology based on regular variation can be used for both, *mono-* and *multi*-fractals, as well as for deterministic and random fractals.

Below, a very brief overview of the main point of the RVT based approach is stated. The formal details and examples, including some astrophysical applications regarding the above mentioned points, will be presented in an extended version of this note (Stern 1997, in preparation).

Let $N_S(\epsilon)$ be the smallest number of balls of the same diameter ϵ, needed to cover a fractal set S. Then $N_S(\epsilon)$ is a *cover function* which provides the *box dimension* of S as $D_B = lim_{\epsilon\to 0}\ln N_S(\epsilon)/\ln(1/\epsilon)$, if the limit exists. The box dimension is the most common form of "fractal dimension" encountered in applications. Denote, $x = 1/\epsilon$; x is a convenient scale variable for the RVT approach. Asymptotically, the power-law form of this cover function, is, $N(x) \approx \wp(x)x^\alpha$, where $\alpha \geq 0$. In general, it is assumed in literature that the various cover functions used to extract scaling exponents are *power-law*-like functions. The main reason why RVT provides a natural framework to study cover functions is that it is a general mathematical theory focusing on asymptotic properties of functions under rescaling transformations of the argument. In particular, RVT provides a general theory of *power-law*-like behavior, including "correction factors" and other fine properties of such functions.

The function $\wp(x)$ has a slower growth than x and is called a *prefactor function.* For simple, self-similar fractals the prefactors, $\wp(x)$, are periodic functions in $\ln(x)$. However, various other forms of prefactor behavior reflecting textural properties of the fractal structure, are possible. This wide spectrum of behaviors can be understood and classified using RVT.

Let $C(x), x \in [x_0, \infty)$ be a generic cover function. The most basic type of behavior of $C(x)$ in RVT is known as *regular variation.* $C(x)$ is *regularly varying at infinity (RV)* if $\lim_{x\to\infty} f(\lambda x)/f(x) = \phi(\lambda)$, where $\phi(\lambda)$ depends only on λ, and has the form $\phi(\lambda) = \lambda^\rho,\ \rho \geq 0$, for any $\lambda > 0$. $C(x)$ is *slowly varying (SV)* if $\phi(\lambda) = 1$, i.e., $\rho = 0$. A slowly varying function cannot grow or decrease "as fast" as a power of x. The exponent ρ is called in RVT, the *regular variation index* of $C(x)$.

The "simplest" fractals, from RVT point of view, have a cover, $C(x)$, which is RV. Such fractals are called here *soft.* For soft fractals ρ coincides with the fractal dimension. Soft fractals are important for applications, since they have "low lacunarity" in the sense that the prefactors are SV functions, and many random fractals of practical interest are likely to be soft, because "randomness" may wash out to some extent strong prefactor oscillations. For soft fractals, a detailed formalism for convergence rates of algorithms for scaling exponents can be provided. The study of fractals with stronger prefactor oscillations than allowed by RV, needs the use of the more involved concepts of *extended regular variation* and *O-regular variation* (see Bingham et al. 1987; Stern 1997, in preparation).

4. A Toy Model for Cosmic Voids Based on Apollonian-like Packing.

The above sketched RVT approach can help to better understand the properties of solid packings of finite regions of 3-D space. The possibility of such a modeling approach to galaxy distributions was suggested by Mandelbrot in his celebrated book (Mandelbrot 1983). Consider, for example, the cover of such a region with nonoverlapping, tangent spheres of different radii ("cosmic voids"). The residual set remaining after removing these spheres is a geometrical construct which is known to be a geometrical multifractal, if the packing is "solid," that is, if the Lebesque measure of the residual set is zero. If the covering spheres are generated deterministically and recursively, the packing is "apollonian" (suggested by Apollon of Perga, 200 BC). Stochastic processes leading to solid random packings of this kind are also possible. The infinite sets of tangency points of the spheres in these constructions form multifractal point sets, providing nontrivial examples of fractal "hierarchical clustering" geometries, potentially relevant to distributions of galaxy counts. These sets can also serve as support for multifractal mass distribution models as well as, a class of generic toy models alternative to more familiar "beta"-type multifractal models common in the galaxy count distributions literature. A visual examination of such tangency point sets suggests that some clustering patterns in such point distributions can mimic, crudely, structures like "walls," "filaments," and "voids" on "all scales," etc. Few generic results exist in the mathematical literature regarding such point sets. For a 3-D apollonian covering with spheres, a crude estimate of the fractal dimension exists (Boyd 1973 gives the value of 2.4). An RVT based approach was undertaken to refine this estimate and investigate the lacunarity of such models, using the properties of a ranked series of radii of the 3-D covering spheres. The formal details and results will be reported elsewhere.

5. Conclusions

The specific texture of a fractal cannot be ignored in the study of fractal-like structures of interest for astrophysics. Textural features carry information about the generating dynamics of a fractal. Information about texture, contained in prefactors of cover functions can be analyzed if the sampling of the structure in "resolution space" is not too sparse. RVT provides a useful formal framework to understand the variety of possible prefactor behaviors and their impact on convergence rates of algorithms for scaling exponents.

A class of multifractal structures based on apollonian-like solid packings of space with spheres may provide an interesting modeling approach to some scaling properties of the large scale distribution of matter. The RVT based methodology, provides some novel ways to study such structures in more detail.

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