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Unified Survey of Fourier Synthesis Methodologies

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Abstract. This paper deals with theoretical and practical results on aperture synthesis. Full attention is devoted to the Fourier synthesis operation, which proves to be the central issue with regard to the global problem. Different regularization techniques are shown to derive from a unique principle: the Principle of Maximum Entropy on the Mean (PMEM). This is the case for WIPE, as well as for the traditional Maximum Entropy Method (and for some others). In order to compare the performances of different regularizers, some numerical simulations are presented.

1. Introduction to Aperture Synthesis

Let $u \equiv u(j,k)$ denote the spatial frequency corresponding to the baseline (j,k) of the interferometric device, and let $\psi(j)$ be the aberration phase term of pupil element j. The central problem is to reconstruct both the brightness distribution ϕ (of some object) and the pupil phase function ψ, by using as data complex-visibility measurements of the form $\mathcal{V}(u(j,k))$. The latter are related to ϕ and ψ by the equation $\mathcal{V}(u(j,k)) = \exp i\beta(u(j,k))\ \widehat{\phi}(u(j,k))$, in which $\beta(u(j,k)) := \psi(j) - \psi(k)$ and $\widehat{\phi}$ is the Fourier transform of ϕ. Solving for ϕ (assuming that β is known) is a Fourier synthesis operation, while solving for β corresponds to a phase calibration operation. The whole problem must be solved in such a way that the solution is not too sensitive to unavoidable measurement errors. In other words, the inverse problem under consideration must be regularized, and it is essential that the chosen methodology provides an estimation of the reconstruction stability. In fact, the Fourier synthesis operation proves to be the core of the problem. It has the form of a linear inverse problem. The corresponding "measurement equation" may be written as $y = Ax$, in which $y \in R^m$ is the real-valued data vector associated with $\mathcal{V}$, A is the Fourier sampling operator, and x is the vector formed with the components of ϕ in an adequate basis (Maréchal & Lannes 1996).

2. Survey of Regularization Techniques

In practice, equation $y = Ax$ fails to have a unique and stable solution, and is therefore replaced by an optimization problem of the form

$$\min_{x \in R^n} \left\{ g(x) := \frac{1}{2}\|y - Ax\|^2 + \alpha f(x) \right\} \tag{1}$$

in which f is a measure of roughness of x, and α is the so-called regularization parameter. The quadratic term in (1) forces the object to fit the data, while the regularization term stabilizes the solution with respect to small variations in y (and of course ensures uniqueness). Denoting by δx the variation of the solution $\bar{x}$ induced by a variation δy, the stability is governed by an inequality such as

$$\|\delta x\| \leq \rho \, \|\delta y\|. \tag{2}$$

The trade-off between fitting the data and stability depends in a crucial manner on the nature of f and on the value of α. We emphasize that the regularization term should also be designed so that g has a physical meaning, so that the solution $\bar{x}$ can be easily interpreted.

Let us now review some classical examples. The well-known Tikhonov regularization corresponds to the case $f(x) = \|x\|^2/2$. It can be generalized by choosing $f(x) = \langle x, Qx\rangle/2$, in which Q is a symmetric non-negative $n \times n$ matrix. When x can be interpreted as a (discrete) probability density, one often takes the Shannon entropy $f(x) = \sum x_j \ln x_j$ or the Kullback measure $f(x) = \sum x_j \ln(x_j/x_{0j})$, in which x_0 represents a prior knowledge of the object. If x is only assumed to be positive, one can use the Generalized Cross-Entropy (GCE) $f(x) = \sum \big(x_j \ln(x_j/x_{0j}) + (x_{0j} - x_j)\big)$. Let us also mention the Itakura-Saito criterion $f(x) = \sum \big(x_j/x_{0j} - \ln(x_j/x_{0j}) - 1\big)$, frequently used in spectral analysis.

Recently, a new Fourier synthesis method has been developed for radio imaging and optical interferometry (Lannes, Anterrieu, & Bouyoucef 1994, 1996; Lannes, Anterrieu, & Maréchal 1997): WIPE. The name of WIPE is associated with that of CLEAN. To some extent (Lannes, Anterrieu, & Maréchal 1997), WIPE can be regarded as an updated version of CLEAN. In particular, the robustness of the reconstruction process is well controlled. The main aspects of WIPE are its regularization principle (for controlling the image resolution) and its matching pursuit strategy (for constructing the image support). The regularizer of WIPE is defined by the relation

$$f(x) = \|Bx\|^2/2 = \langle x, B^T Bx\rangle/2, \tag{3}$$

in which $\|Bx\|^2$ represents the energy of x in the high frequency band (Lannes, Anterrieu, & Bouyoucef 1994). Clearly, this regularization principle belongs to the Tikhonov family, since $B^T B$ is a symmetric non-negative $n \times n$ matrix. It is very closely related to the notion of resolution. The function to be minimized in (1) takes the form $\|y_0 - A_0 x\|^2/2$ with $A_0 = [A; B]$ and $y_0 = (y; 0)$. This kind of function is efficiently minimized by a conjugate-gradients algorithm, which has the advantage of providing (with negligible additional computing time) an estimate of the condition number and, thereby, control of the stability of the reconstruction process. As for the matching pursuit strategy, we simply mention that the main difference from CLEAN is that it can be conducted at the level of the scaling functions of the object workspace (Lannes, Anterrieu, & Bouyoucef 1994)

3. Unification Results

We now turn to the unification of WIPE with the methodologies mentioned in §2 by the Principle of Maximum Entropy on the Mean. Note first that the optimization problem (1) is equivalent to

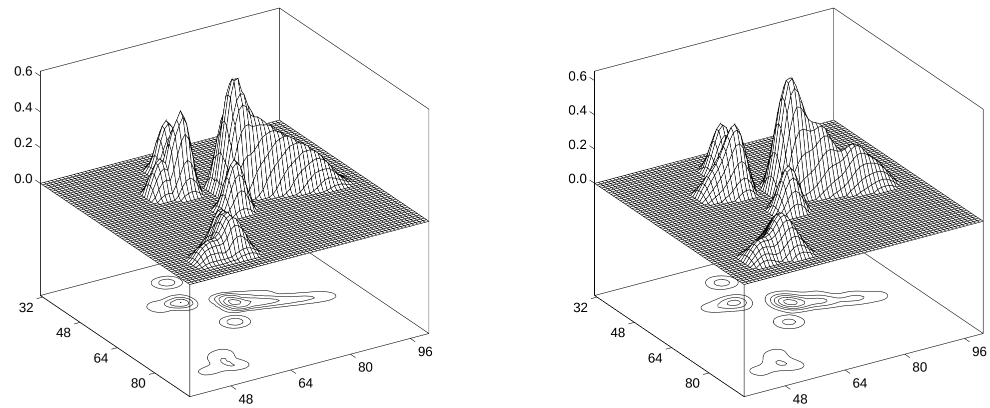
$$\min_{(x;b)\in R^{n+m}} \left\{ \frac{1}{2}\|b\|^2 + \alpha\, f(x) \;\middle|\; y - Ax = b \right\}$$

in which we have explicitly introduced the error term b. In the PMEM, $(x; b)$ is a random vector to which a prior probability measure $\mu \otimes \nu$ is assigned. The Maximum Entropy Principle is then used to infer a posterior probability on $(x; b)$, and finally, we choose the expectancy of x under the inferred density $\bar{p}$ as the solution to our inverse problem. The core of the PMEM is an infinite dimensional linearly constrained optimization problem (Maréchal & Lannes 1996) in which the functional to be minimized is (the continuous version of) the Kullback information measure. As explicitly shown in Maréchal & Lannes (1996), it can be solved by means of a dual strategy. It is then possible to demonstrate that, for particular choices of the priors μ and ν, the PMEM gives rise to some of the most classical regularization techniques. For example, a Gaussian ν with αI as the covariance matrix, associated with a Gaussian μ with covariance Q, gives rise to a Tikhonov regularization technique. In particular, if $Q = (B^T B)^{-1}$ (provided that $B^T B$ is positive definite), we retrieve Wipe. Now, taking the multidimensional Poissonian distribution with vector parameter x_0 as μ yields the GCE regularizer (Lannes, Anterrieu, & Bouyoucef 1994), and the Gamma law with vector parameter x_0 gives rise to the Itakura-Saito criterion. Note that in this description, the quadratic fit term derives from the standard Gaussian prior measure on b. If another kind of noise were to corrupt the data, this prior may, of course, be replaced by the appropriate one.

Many other criteria could be derived from the above scheme, taking into account the probabilistic description of the problem. However, it is important to keep in mind that the design of f (i.e., the choice of the corresponding μ) must be governed by the nature of the imaging operator A. The next paragraph illustrates the importance of this point.

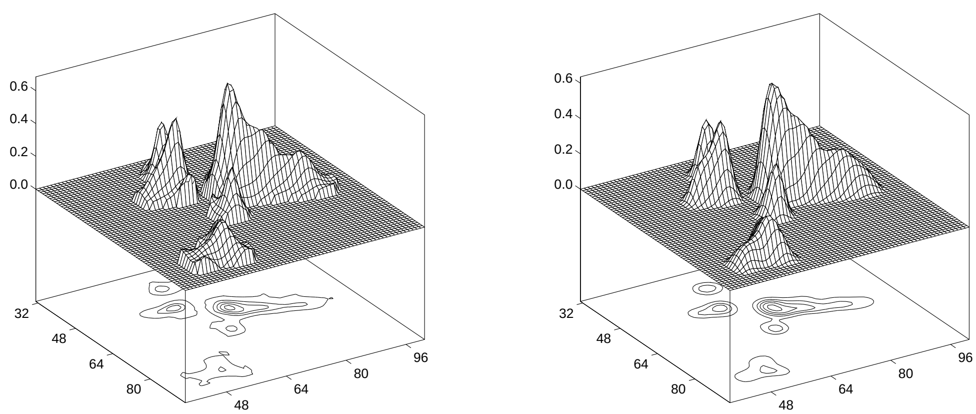
4. Simulations and Conclusion

In the simulations presented here, the frequency coverage consists of 211 points. The object to be reconstructed, shown in Figure 1(a), is the original object convolved by a point-spread function corresponding to the selected resolution limit (i.e., to the corresponding frequency coverage to be synthesized; see Lannes, Anterrieu, & Bouyoucef 1994). Three reconstructions were performed: with Wipe (Figure 1(b)), the Shannon entropy on the support determined by the matching pursuit strategy of Wipe (Figure 2(a)), and the GCE in which the prior model x_0 is a smoothed (and normalized) version of the characteristic function of the previous support (Figure 2(b)). For the Shannon entropy and the GCE, the regularization parameter α was adjusted in such a way that the final fit term is equal to that reached by Wipe. The best reconstructed images, shown in Figures 1(b) and 2(b), are quite similar, the values of the stability parameter ρ being reasonably small. In both cases, the inverse problem is well regularized. However, for the selected resolution, the best stability is obtained with the Wipe regularizer.



(a) (b)

Figure 1. a: image to be reconstructed; b: reconstructed image by WIPE ($\rho = 2.96$).



(a) (b)

Figure 2. Reconstruction via the GCE; a: with a uniform prior x_{0j} on the support provided by WIPE ($\rho = 4.41$); b: with a smoothed version of the previous prior ($\rho = 3.28$).

References

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