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A Theoretical Photometric and Astrometric Performance Model for Point Spread Function CCD Stellar Photometry

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Abstract. Using a simple 2-D Gaussian Point Spread Function (PSF) on a constant (flat) sky background, I derive a theoretical photometric and astrometric performance model for analytical and digital PSF-fitting stellar photometry. The theoretical model makes excellent predictions for the photometric and astrometric performance of over-sampled and under-sampled CCD stellar observations even with cameras with pixels that have large *intra*-pixel quantum efficiency variations. The performance model accurately predicts the photometric and astrometric performance of realistic space-based observations from segmented-mirror telescope concepts like the Next Generation Space Telescope with the MATPHOT algorithm for digital PSF CCD stellar photometry which I presented last year at ADASS XI. The key PSF-based parameter of the theoretical performance model is the effective background area which is defined to be the reciprocal of the volume integral of the square of the (normalized) PSF; a critically-sampled PSF has an effective background area of 4π (≈ 12.57) pixels. A bright star with a million photons can theoretically simultaneously achieve a signal-to-noise ratio of 1000 with a (relative) astrometric error of a *milli*pixel. The photometric performance is maximized when either the effective background area or the effective-background-level measurement error is minimized. Real-world considerations, like the use of poor CCD flat fields to calibrate the observations, can and do cause many existing space-based and ground-based CCD imagers to fail to live up to their theoretical performance limits. Future optical and infrared imaging instruments can be designed and operated to avoid the limitations of some existing space-based and ground-based cameras. This work is supported by grants from the Office of Space Science of the National Aeronautics and Space Administration (NASA).

1. Photometry

1.1. Bright Star Limit

Let us assume that the variance of the noise associated with the i^{th} pixel of an observation of a bright star is due only to stellar photon noise,

$$\sigma_i^2 \equiv S_E \Phi(x_i, y_i) \,,$$

where Φ is the normalized *sampled* Point Spread Function (PSF). All other noise sources (for example, the background sky, instrumental readout noise, etc.) are assumed, in this case, to be negligibly small. The variance of the stellar intensity measurement of bright *over-sampled* stars is thus

$$\sigma^2_{S_E:\,\mathrm{bright}} \approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\frac{\partial}{\partial S_E} S_E \Phi(x_i,y_i)\right)^2\right]^{-1} \equiv \left[\sum_{i=1}^{N} \frac{1}{S_E\Phi(x_i,y_i)}\Phi^2(x_i,y_i)\right]^{-1}$$

$$= \left[\frac{1}{S_E}\sum_{i=1}^{N}\Phi(x_i,y_i)\right]^{-1} \approx S_E\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi(x,y)\,dx\,dy\right]^{-1} \equiv S_E\,,$$

as expected from photon statistics with a normalized unsampled PSF (ϕ).

1.2. Faint Star Limit

Let us assume that we can replace the measurement error associated with the i^{th} pixel of an observation of a faint star with an average *constant* rms value of

$$\sigma^2_{\mathrm{rms}} \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{-2} \approx B_E + \sigma^2_{\mathrm{RON}},$$

where B_E is the constant background level in electrons per pixel (e^-/px) and σ^2_{RON} is the square of the rms readout noise (e^-/px). Using this approximation, we find that the variance of the stellar intensity measurement of faint *over-sampled* stars is

$$\sigma^2_{S_E:\,\mathrm{faint}} \approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\frac{\partial}{\partial S_E} S_E \Phi(x_i,y_i)\right)^2\right]^{-1} \equiv \left[\sum_{i=1}^{N} \frac{1}{\sigma^2_{\mathrm{rms}}}\Phi^2(x_i,y_i)\right]^{-1}$$

$$= \sigma^2_{\mathrm{rms}}\left[\sum_{i=1}^{N}\Phi^2(x_i,y_i)\right]^{-1} \approx \sigma^2_{\mathrm{rms}}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi^2(x,y)\,dx\,dy\right]^{-1} \equiv \sigma^2_{\mathrm{rms}}\,\beta\,,$$

where the constant β is the "effective background area" defined as the *reciprocal* of the volume integral of the *square* of the normalized unsampled PSF (ϕ). The effective background area for a normalized Gaussian PSF with a standard deviation of S_σ px is $\beta = 4\pi S_\sigma^2$ px^2; the value for a critically-sampled normalized Gaussian is, by definition, 4π ($\approx$12.57) px^2. King (1983) identifies β as the "equivalent-noise area" and notes that numerical integration of a realistic ground-based stellar profile gives an equivalent area of $30.8 S_\sigma^2$ px^2 instead of the value of $4\pi S_\sigma^2$ px^2 for a Gaussian profile.

1.3. Photometic Performance Model

A simple performance model for photometry can be created by combining the bright and faint star limits developed above. The total variance of the stellar intensity measurement of over-sampled stars is thus

$$\sigma^2_{S_E} \approx \sigma^2_{S_E:\,\mathrm{bright}} + \sigma^2_{S_E:\,\mathrm{faint}} = S_E + \sigma^2_{\mathrm{rms}}\,\beta \approx S_E + \beta\left[B_E + \sigma^2_{\mathrm{RON}}\right]\ .$$

The term in brackets in the last equation can physically be thought of as the "effective background level".

An important, but frequently ignored, noise source is the uncertainty of the measurement of the effective background level ($\sigma_{\rm B}$). If the "sky" background is assumed to be flat, then the lower limit for measurement error of the effective background level is

$$\sigma_{\rm B} \lesssim \frac{\sqrt{B_E + \sigma_{\rm RON}^2}}{\sqrt{N}} \,.$$

In order to have a more realistic performance model for photometry, this noise source must be added as the square of $\beta\sigma_{\rm B}$ because it is a systematic error:

$$\sigma_{S_E} \approx \sqrt{S_E + \beta\left[B_E + \sigma_{\rm RON}^2\right] + (\beta\sigma_{\rm B})^2} \;\; {\rm e}^- \,.$$

Photometric performance will be maximized when either the effective background area (β) or the effective-background-level measurement error (σ_B) is minimized.

We now have the basis for a simple, yet realistic, photometric performance model for PSF-fitting algorithms. An upper limit for the theoretical signal-to-noise ratio of a PSF-fitting algorithm is

$$\boxed{\mathit{SNR} \lesssim \frac{S_E}{\sqrt{S_E + \beta\left[B_E + \sigma_{\rm RON}^2\right] + (\beta\sigma_{\rm B})^2}} \,.}$$

2. Astrometry

2.1. Bright Star Limit

Let us again assume that the variance of the noise associated with the $i^{\rm th}$ pixel of an observation of a bright star is due only to stellar photon noise. The variance of the stellar x position measurement, S_X, of bright *over-sampled* stars with a normalized *unsampled* Gaussian PSF at the $i^{\rm th}$ pixel is

$$\phi_i \equiv \phi(x_i, y_i; S_X, S_Y, S_\sigma) \equiv \frac{1}{2\pi S_\sigma^2} \exp\left(-\frac{(x-S_X)^2 + (y-S_Y)^2}{2S_\sigma^2}\right) \,,$$

is

$$\begin{aligned}
\sigma^2_{S_X\text{: bright}} &\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\frac{\partial}{\partial S_X} S_E\phi_i\right)^2\right]^{-1} \equiv \left[\sum_{i=1}^{N} \frac{1}{S_E\phi_i}\left(S_E\phi_i \frac{S_X - x_i}{S_\sigma^2}\right)^2\right]^{-1} \\
&\approx \left[\frac{S_E}{S_\sigma^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi(x,y)(S_X - x)^2\, dx\, dy\right]^{-1} = \frac{S_\sigma^4}{S_E}\left[S_\sigma^2\right]^{-1} \equiv \frac{1}{S_E}\left(\frac{\beta}{4\,\pi}\right) ,
\end{aligned}$$

where β is the effective background area as defined above. By symmetry, the variance of the stellar y position measurement of bright over-sampled stars is the same.

2.2. Faint Star Limit

Let us again assume that noise contribution from the star is negligibly small and that can replace σ_i^2 with with an average *constant* rms value of $\sigma^2_{\rm rms}$. Using this approximation, we find that the variance of the stellar x position measurement of faint *over-sampled* stars with a normalized *unsampled* Gaussian PSF is

$$\begin{aligned}
\sigma^2_{S_X:\,\rm faint} &\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(\frac{\partial}{\partial S_X} S_E \phi_i\right)^2\right]^{-1} \equiv \left[\sum_{i=1}^{N} \frac{1}{\sigma^2_{\rm rms}} \left(S_E \phi_i \frac{S_X - x_i}{S_\sigma^2}\right)^2\right]^{-1} \\
&\approx \sigma^2_{\rm rms} \left[\frac{S_E^2}{S_\sigma^4} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi^2(x,y)(S_X - x)^2\, dx\, dy\right]^{-1} = \sigma^2_{\rm rms} \frac{S_\sigma^4}{S_E^2} \left[\frac{1}{8\,\pi}\right]^{-1} \\
&\equiv \sigma^2_{\rm rms} \frac{\beta^2}{2\,\pi\, S_E^2} \equiv 8\,\pi\,\sigma^2_{\rm rms} \left(\sigma^2_{S_X:\,\rm bright}\right)^2 .
\end{aligned}$$

By symmetry, the variance of the stellar y position measurement of faint over-sampled stars is the same.

2.3. Astrometric Performance Model

We can now create a simple performance model for astrometry by combining the bright and faint star limits developed above. The expected lower limit of the rms measurement error of the stellar x position of a PSF-fitting algorithm is

$$\boxed{\begin{aligned}
&\sigma_{S_X} \\
&\approx \sqrt{\sigma^2_{S_X:\,\rm bright} + \sigma^2_{S_X:\,\rm faint}} \\
&\gtrsim \sqrt{\left[\left(\frac{\beta}{4\,\pi}\right)\frac{1}{S_E}\right]\left[1 + \left(\frac{\beta}{4\,\pi}\right)\frac{8\,\pi}{S_E}(B_E + \sigma^2_{\rm RON})\right]}\ \text{px}\,.
\end{aligned}}$$

By symmetry, the expected lower limit of the variance of the measured y coordinate of the stellar position of a PSF-fitting algorithm is the same.

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References

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